

Having Solution or Not

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Learning Target

Review



$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

System of Linear Equations

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

- Given A and \mathbf{b} , sometimes \mathbf{x} exists (having solution), and sometimes doesn't (no solution)
- New terms: “*linear combination*” and “*span*”

Solution

- Given a system of linear equation

$$\begin{array}{rcl} 2x_1 - 3x_2 + x_3 & = & -10 \\ x_1 + x_3 & = & 3 \end{array} \quad \begin{array}{c} \vdots \\ \end{array}$$

$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ is a solution

$\begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$ is also a solution

There are other possible solutions

The **set** of **all solutions** of a **system of linear equations** is called the **solution set**.

Solution

- A system of linear equations is called **consistent** if it has one or more solutions.
- A system of linear equations is called **inconsistent** if its solution set is empty.

| | | | |
|-----------------------------------|---|---|--|
| | $\begin{aligned} 3x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= 0 \end{aligned}$ | $\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 20 \end{aligned}$ | $\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$ |
| Solution set | $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ | $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix} : \forall t \in \mathcal{R} \right\}$ | $\{\}, \text{ or } \phi$ |
| Consistent or Inconsistent? | Consistent | Consistent | Inconsistent |

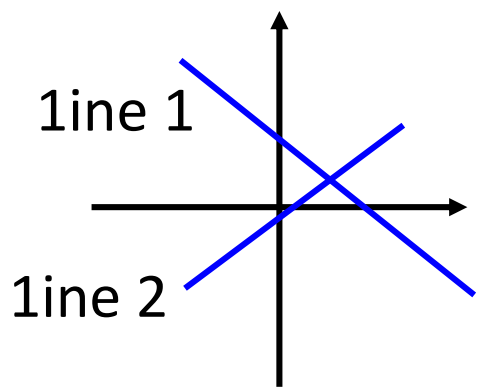
Solution (High School)

More
Variables?

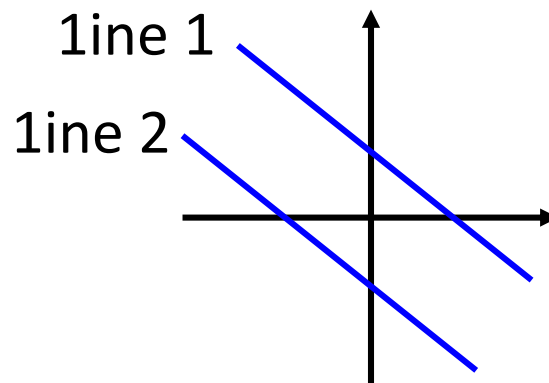
- Considering any system of linear equations with 2 variables and 2 equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{..... line 1}$$

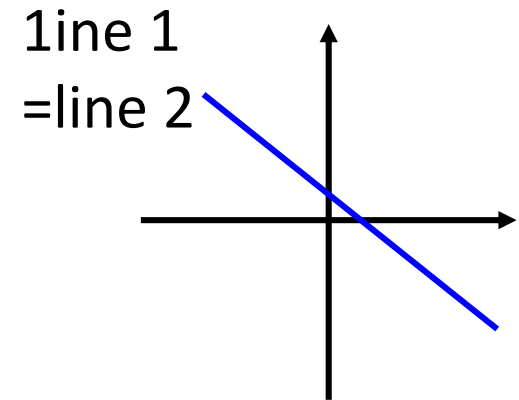
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \text{..... line 2}$$



unique solution



no solution



infinitely
many solution

Does a system of linear equations have solutions?

Linear Combination

Linear Combination

- Given a vector set $\{u_1, u_2, \dots, u_k\}$
- The linear combination of the vectors in the set:
 - $v = c_1u_1 + c_2u_2 + \dots + c_ku_k$
 - c_1, c_2, \dots, c_k are scalars (Coefficients of linear combination)

vector set: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

coefficients: $\{-3, 4, 1\}$

What is the result of linear combination?

Column Aspect

$$\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

$a_1 \quad a_2 \quad \quad \quad a_n$

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

Vector set

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

coefficients

$$A\mathbf{x} = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

Linear
Combination

System of Linear Equations v.s. Linear Combination

$$A\mathbf{x} = \mathbf{b}$$

(A system of linear equations)

Non empty solution set?

Has solution or not?

Consistent?

The Same question

Column Aspect

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

the linear combination of columns of A

Is \mathbf{b} the linear combination of columns of A ?

Example 1

$$\begin{aligned} 3x_1 + 6x_2 &= 3 \\ 2x_1 + 4x_2 &= 4 \end{aligned}$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$
$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Is \mathbf{b} the linear combination of columns of A ?

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$$

Example 1

$$3x_1 + 6x_2 = 3$$

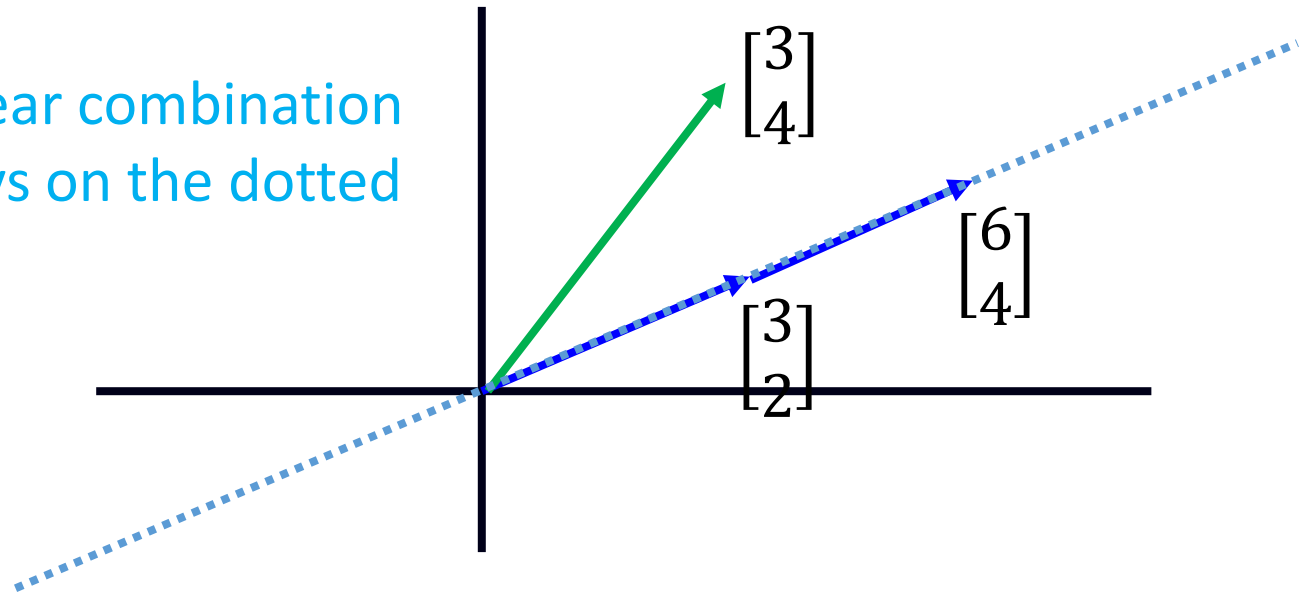
$$2x_1 + 4x_2 = 4$$

Has solution or not?

• Vector set: $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$

• Is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$? No

The linear combination is always on the dotted line.



Example 2

$$2x_1 + 3x_2 = 4$$

$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Is \mathbf{b} the linear combination of columns of A ?

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

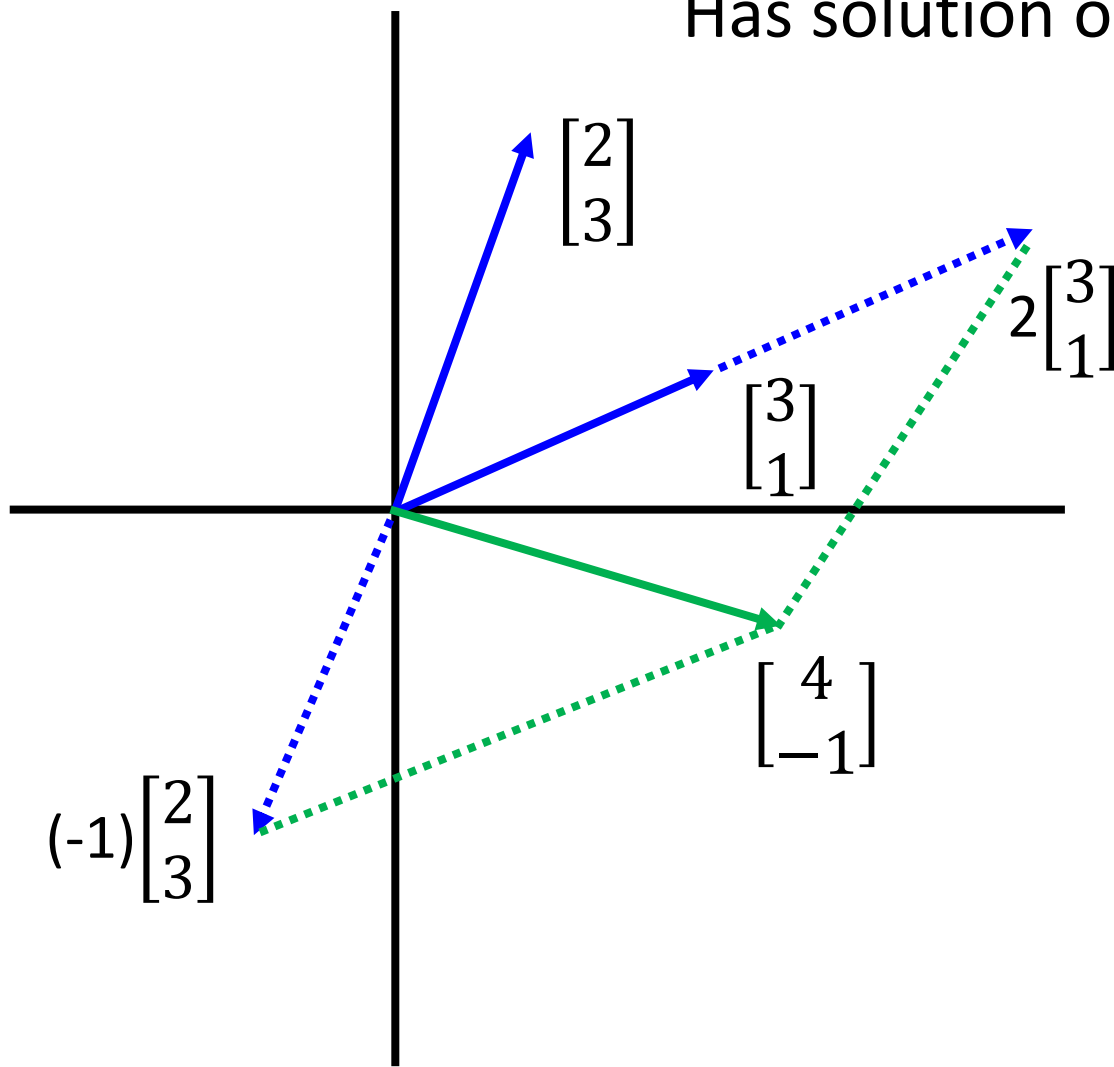
Example 2

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 3x_1 + 1x_2 &= -1 \end{aligned}$$

Has solution or not?

$$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

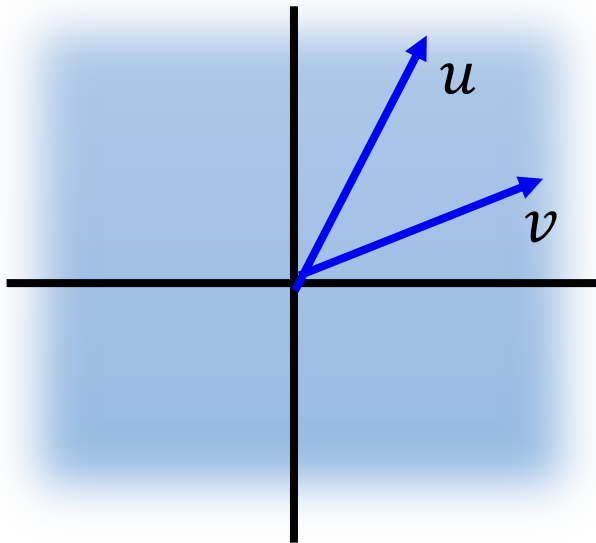
$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$



Example 2

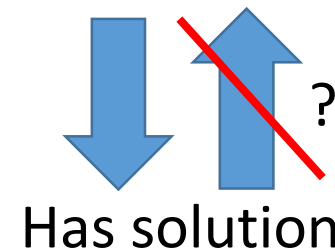
How about in \mathcal{R}^4 ?

- If \mathbf{u} and \mathbf{v} are any nonparallel vectors in \mathcal{R}^2 , then every vector in \mathcal{R}^2 is a linear combination of \mathbf{u} and \mathbf{v}
 - Nonparallel: \mathbf{u} and \mathbf{v} are nonzero vectors, and $\mathbf{u} \neq c\mathbf{v}$.



$$\begin{aligned}u_1x_1 + v_1x_2 &= b_1 \\u_2x_1 + v_2x_2 &= b_2\end{aligned}$$

\mathbf{u} and \mathbf{v} are not parallel



- If \mathbf{u} , \mathbf{v} and \mathbf{w} are any nonparallel vectors in \mathcal{R}^3 , then every vector in \mathcal{R}^3 is a linear combination of \mathbf{u} , \mathbf{v} and \mathbf{w} ?

Example 3

$$2x_1 + 6x_2 = -4$$

$$1x_1 + 3x_2 = -2$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Is \mathbf{b} the linear combination of columns of A ?

$$\begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$$

Example 3

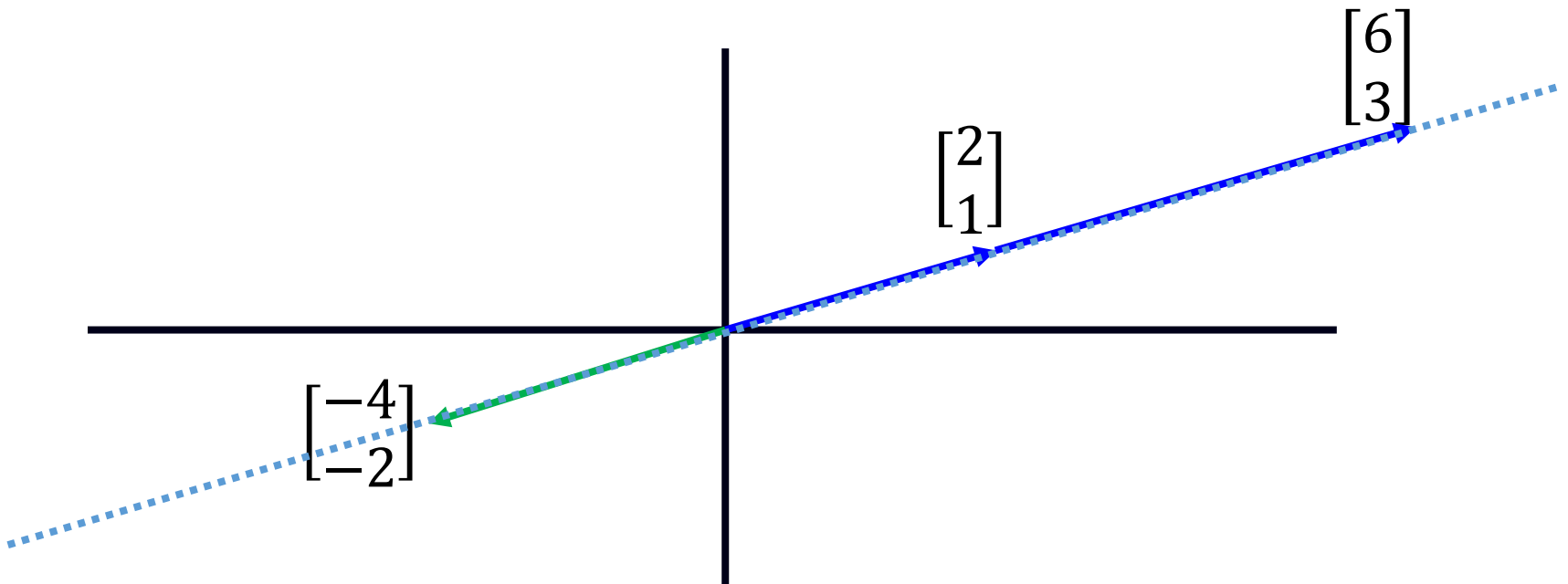
$$2x_1 + 6x_2 = -4$$

$$1x_1 + 3x_2 = -2$$

Has solution or not?

- Vector set: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$

- Is $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$? Yes



Does a system of linear equations have solutions?

Span

Span

- A vector set $S = \{u_1, u_2, \dots, u_k\}$
- Span of S is the vector set of all linear combinations of u_1, u_2, \dots, u_k
 - Denoted by $\text{Span} \{u_1, u_2, \dots, u_k\}$ or $\text{Span } S$
 - $\text{Span } S = \{c_1u_1 + c_2u_2 + \dots + c_ku_k \mid \text{for all } c_1, c_2, \dots, c_k\}$
- Vector set $V = \text{Span } S$
 - “ S is a generating set for V ” or “ S generates V ”
 - A vector set generated by another vector set is called **Space**

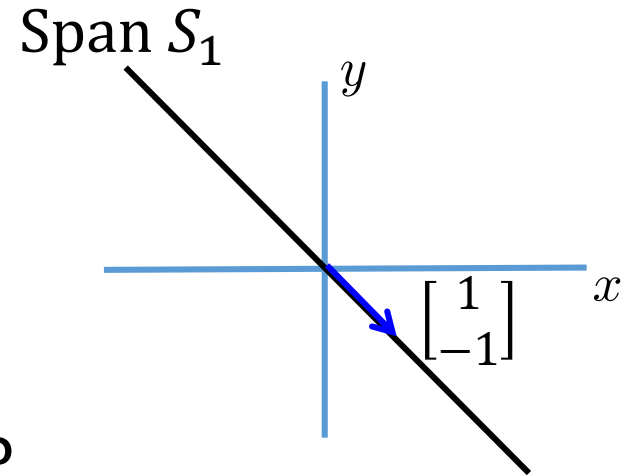
Span

- Let $S_0 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, what is $\text{Span } S_0$?

- Ans: $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ (only one member)

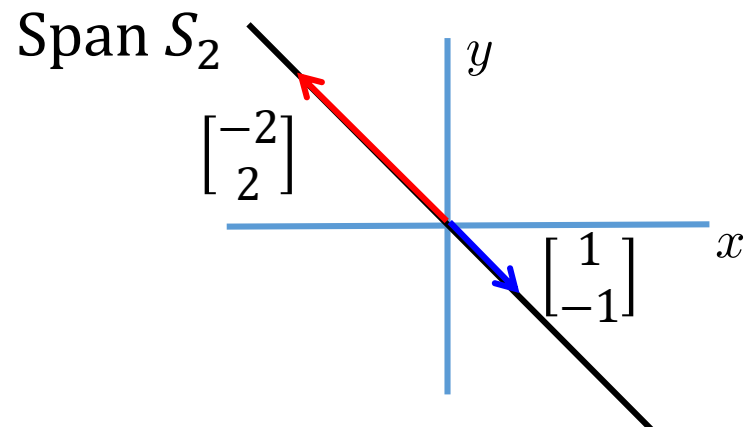
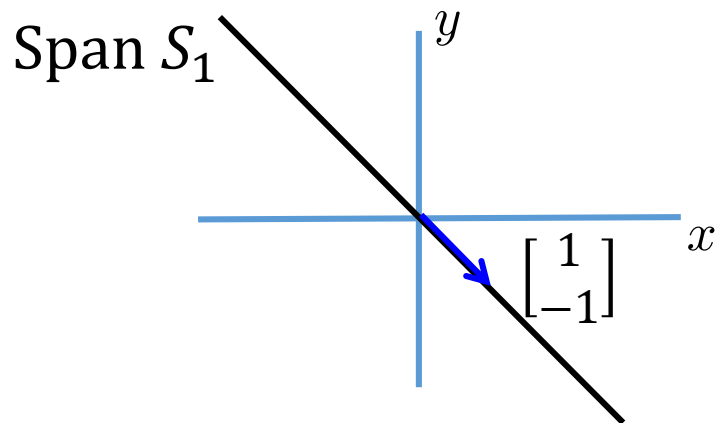
- Let $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, what is $\text{Span } S_1$?

- If S contains a non zero vector, then $\text{Span } S$ has infinitely many vectors



Span

- Let $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, what is $\text{Span } S_1$?
- Let $S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$, what is $\text{Span } S_2$?



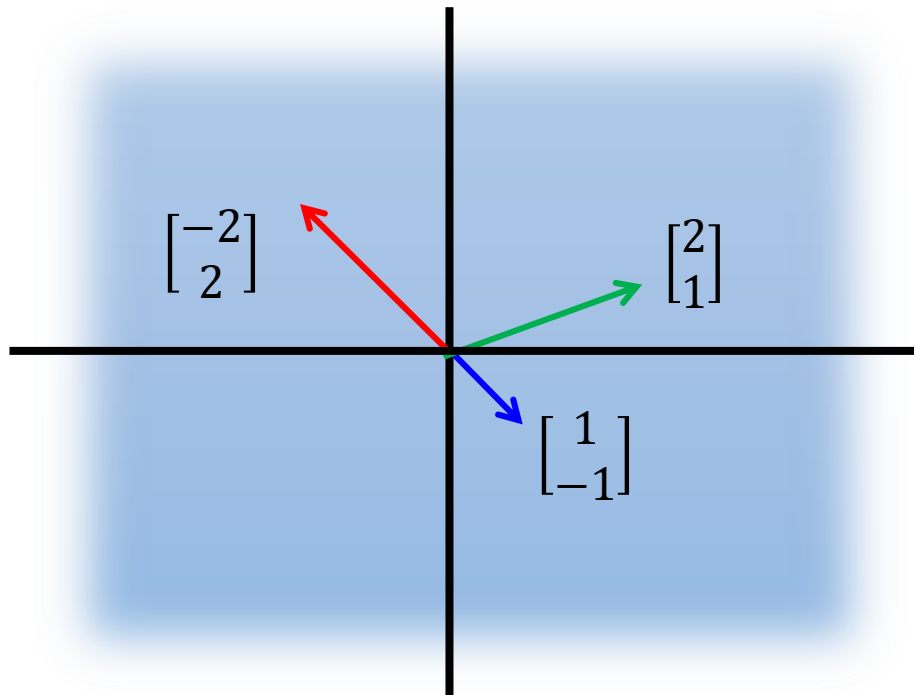
$$\text{Span } S_1 = \text{Span } S_2$$

(Different number of vectors can generate the same space.)

Span

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is $\text{Span } S_3$?

nonparallel vectors

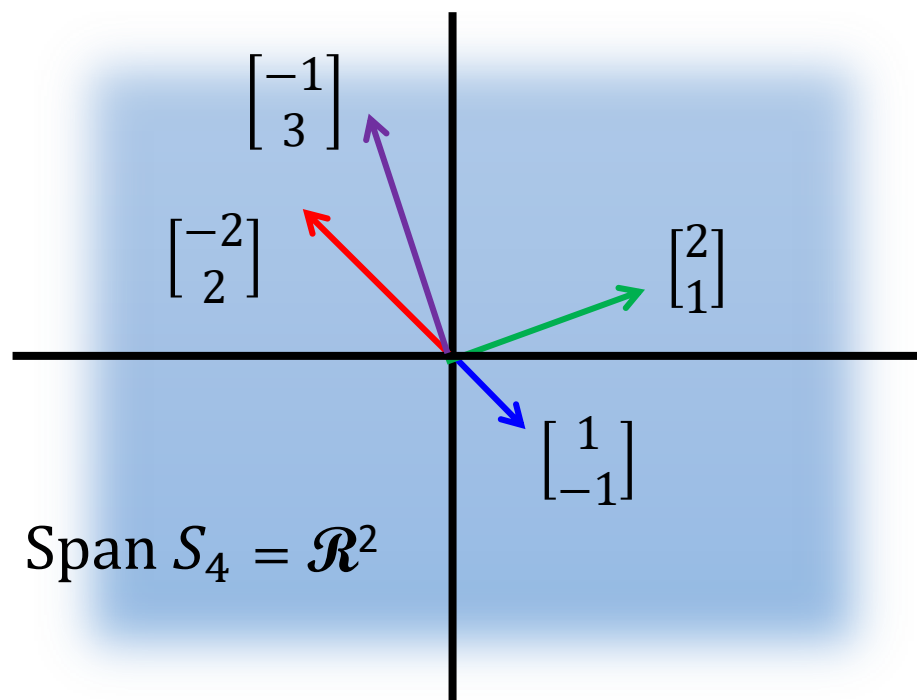
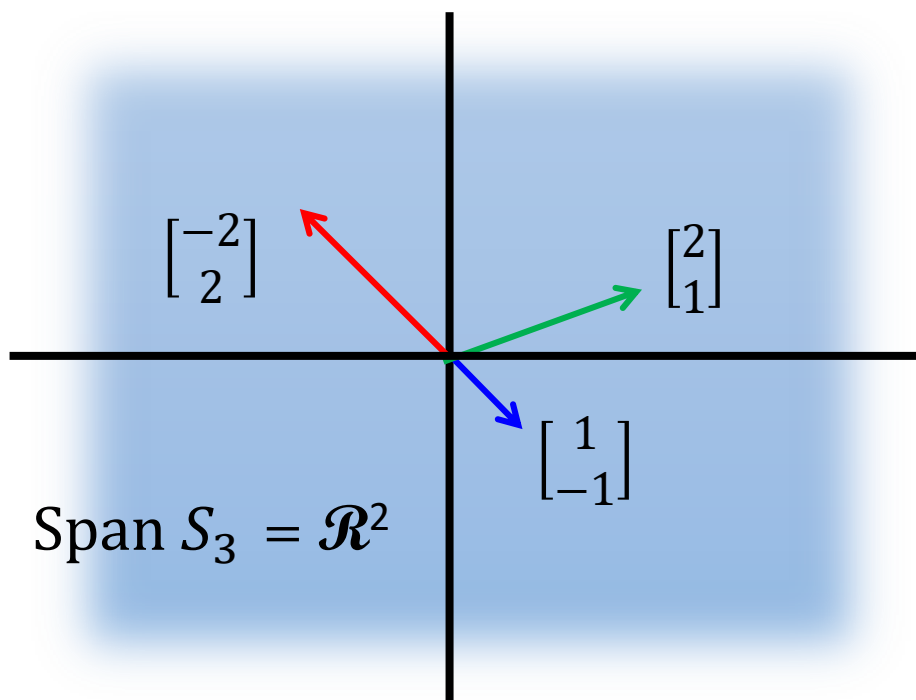


Every vector in \mathcal{R}^2
is their linear
combination

$$\text{Span } S_3 = \mathcal{R}^2$$

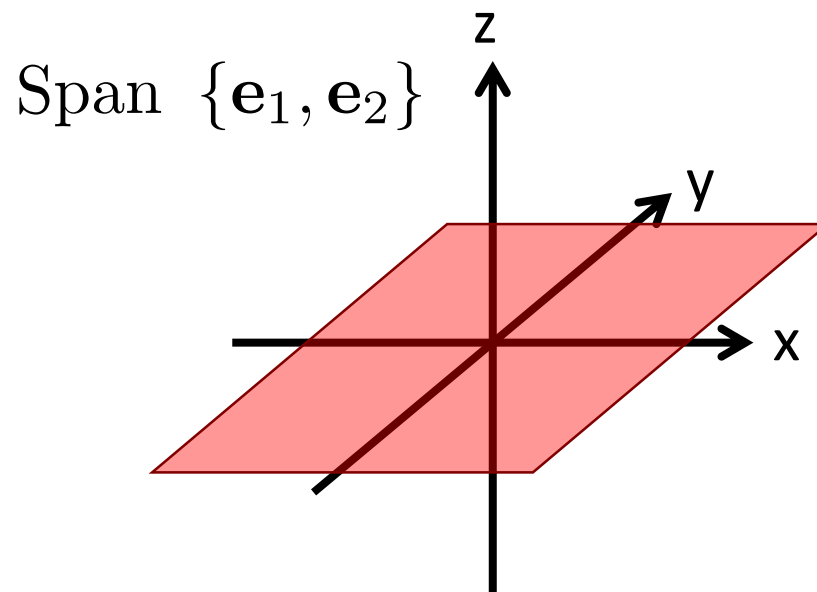
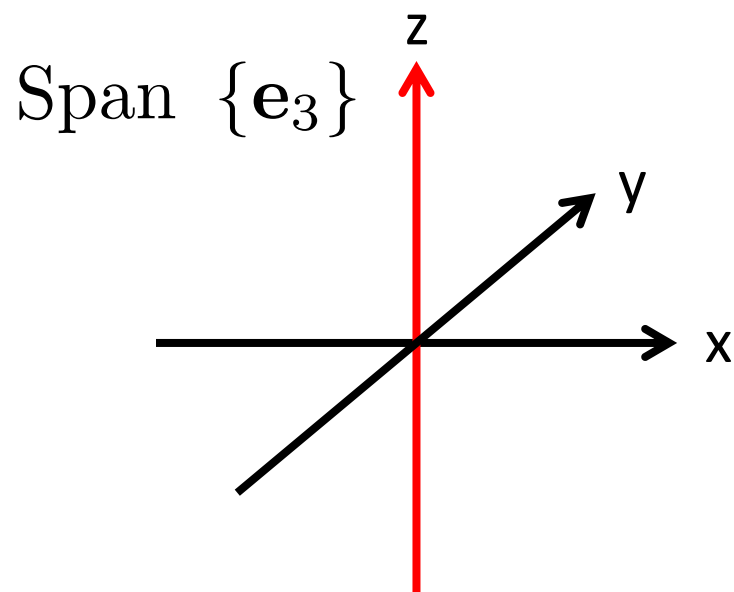
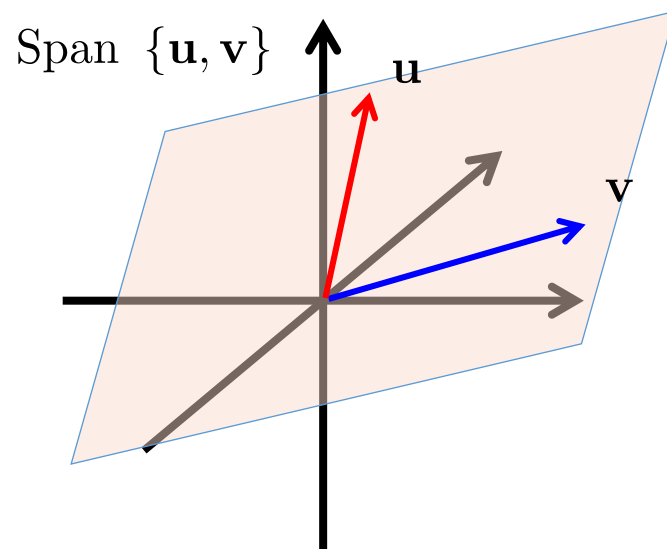
Span

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is $\text{Span } S_3$?
- Let $S_4 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$, what is $\text{Span } S_4$?



Span

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
 \end{aligned}$$

$$A\mathbf{x} = \mathbf{b}$$

Has solution or not?



The same question

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Is b the linear combination of columns of A ?



The same question

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

Is b in the span of the columns of A ?

Summary

$$A\mathbf{x} = \mathbf{b}$$

Does a system of linear equations have solution?

Is b a linear combination of columns of A ?

Is b in the span of the columns of A ?

YES → Have solution

NO → No solution